Math 565: Functional Analysis
Lecture 1

Complex numbers. $\mathbb{C} := \mathbb{R}[i]/_{i^2+1}$, i.e. $\mathbb{C} = \{a+ib : a, b \in \mathbb{R}\}$ where $i^2 = -1$, so $(a_1+ib_1)(a_2+ib_2) = (a_1a_2 - b_1b_2) + i(b_1a_2 + a_1b_2)$ and $(a_1+ib_1)+(a_2+ib_2) = (a_1+a_2) + i(b_1+b_2)$. As a vector space over $\mathbb{R}$, $\underbrace{\mathbb{C} \text{ is isomorphic to } \mathbb{R}^2}$.

We equip $\mathbb{C}$ with a norm: $|a+ib| = \sqrt{|a|^2+|b|^2}$, hence isomorphic to the 2-norm on $\mathbb{R}^2$, hence $\mathbb{C}$ is complete in this norm.

For $\alpha \in \mathbb{C}$, $\alpha = a+ib$, denote

- $\operatorname{Re}\alpha := a$ and $\operatorname{Im}\alpha := b$.
- $\bar{\alpha} := a - ib$, called the conjugate of α. Note that $\alpha\bar{\alpha} = a^2+b^2 = |\alpha|^2$.
- $\operatorname{sgn}(\alpha) := \frac{\alpha}{|\alpha|}$ so $|\alpha| = \frac{\alpha\bar{\alpha}}{|\alpha|} = \alpha \cdot \operatorname{sgn}(\bar{\alpha})$ for $\alpha \neq 0$.
- $\alpha = r \cdot e^{2\pi i\theta}$, where $r := |\alpha| \geq 0$ and $e^{2\pi i\theta} = \operatorname{sgn}(\alpha)$.

Fundamental Thm of Algebra. Every polynomial $p \in \mathbb{C}[x]$ has a root in $\mathbb{C}$. In particular, $p(x) = \prod_{i=1}^{n}(x - \alpha_i)$, where $n := \deg(p)$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ are the roots of $p(x)$.

Banach spaces.

Let X be a $\mathbb{C}$-vector space. A seminorm on X is a function $\|\cdot\| : X \to [0,\infty)$ satisfying:

(i) $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|$ for all $\alpha \in \mathbb{C}$, $x \in X$.

(ii) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$. (Implies: $|\|x\| - \|y\|| \leq \|x-y\|$.)

If a seminorm also satisfies (iii) $\|x\| = 0 \Rightarrow x = 0$, then we call it a norm.

A normed vector space (over $\mathbb{C}$) is a pair $(X, \|\cdot\|)$ where X is a $\mathbb{C}$-vector space and $\|\cdot\|$ is a norm on X.

As usual, a norm $\|\cdot\|$ defines a metric on X by
$$d(x,y) := \|x-y\|.$$
We call the topology of this metric the norm topology on X, and convergence in this topology/metric is called convergence in norm.
If this metric is complete, we call $(X, \|\cdot\|)$ a Banach space.

Examples.

(a) $\mathbb{C}^d$, $1 \le d < \infty$, is a Banach space equipped with the p-norm for any $1 \le p \le \infty$, i.e. for $\vec{x} \in \mathbb{C}^d$, $\|\vec{x}\|_p := \left(\sum_{i=1}^{d} |x_i|^p\right)^{\frac{1}{p}}$ if $p < \infty$ and

$$\|\vec{x}\|_\infty := \max_{1 \le i \le d} |x_i|.$$

All these norms are Lipschitz-equivalent, i.e. for any $1 \le p, q \le \infty$, $\exists\, c, C > 0$ such that
$$c\|x\|_q \le \|x\|_p \le C \cdot \|x\|_q.$$

(b) For a measure space $(X, \mathcal{B}, \mu)$, let $L^1(\mu) := L^1(X, \mathcal{B}, \mu)$ denote the quotient of the vector space of all μ-integrable functions by the equivalence relation $f =_\mu g$ (i.e. $f = g$ μ-a.e.).
We showed last semester that this is a Banach space, using the absolutely convergent series criterion.
We will discuss $L^p(\mu)$ spaces, $1 \le p \le \infty$, later.

(c) Let X be a set. Denote by $B(X)$ the set of all bdd $\mathbb{C}$-valued functions on X. Then $B(X)$ is a $\mathbb{C}$-vector space under pointwise addition and scalar multiplication. We equip this space with the sup norm, namely:

$$\|f\|_{sup} := \sup_{x\in X} |f(x)|.$$

It follows from the Δ-ineq. for $|\cdot|$ on $\mathbb{C}$ that $\|\cdot\|_{sup}$ is indeed a norm.

Claim. $B(X)$ with sup norm is a Banach space.

Proof. Let $(f_n) \subseteq B(X)$ be a norm-Cauchy sequence. Then for each $x \in X$, $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_{sup}$, hence the sequence $(f_n(x)) \subseteq \mathbb{C}$ is Cauchy, so it converges: $f(x) := \lim_{n\to\infty} f_n(x)$. Furthermore, note that

$$\big|\|f_n\|_{sup} - \|f_m\|_{sup}\big| \leq \|f_n - f_m\|_{sup}$$

hence the sequence $(\|f_n\|_{sup})_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is also Cauchy, thus converges to some $c \in [0,\infty)$. But then for each $x \in X$,
$|f(x)| = \lim_{n\to\infty} |f_n(x)| \leq \lim_{n\to\infty} \|f_n\| = c$, so $\|f\|_{sup} \leq c$ hence $f \in B(X)$.

Lastly, we need to show that $\|f - f_n\|_{sup} \xrightarrow[\infty]{n} 0$, i.e. $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n \geq N$, we have $\|f - f_n\|_{sup} \leq \varepsilon$, equivalently,
$|f(x) - f_n(x)| \leq \varepsilon$ for all $x \in X$.
But for each fixed $x \in X$, we have
$$|f(x) - f_n(x)| \leq |f(x) - f_{n+k}(x)| + |f_{n+k}(x) - f_n(x)| \leq |f(x) - f_{n+k}(x)| + \|f_{n+k} - f_n\|_{sup}$$
so $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $\|f_{n+k} - f_n\|_{sup} < \varepsilon/2$ by the norm-Cauchyness. But also we can k large enough (depending on the specific x and ε) so that $|f(x) - f_{n+k}(x)| < \varepsilon/2$. Hence
$|f(x) - f_n(x)| \leq \varepsilon/2 + \varepsilon/2 \leq \varepsilon$ for all $n \geq N$. Thus, $\|f - f_n\|_{sup} \leq \varepsilon$. □

Observation. Closed subspaces of Banach spaces are Banach spaces.

This gives more examples:

(d) Let X be a topological space and denote by $BC(X)$ the set of bounded continuous $\mathbb{C}$-valued functions on X. Observe that $BC(X)$ is a subspace of the vector space $B(X)$. We show that $BC(X)$ is a Banach space by proving:

Claim. $BC(X)$ is closed in $B(X)$.

Proof. Let $f_n \to f$ in $B(X)$ where $(f_n) \subseteq BC(X)$. We show that $f \in BC(X)$ as well. We fix $x_0 \in X$ and show that f is continuous at x_0, i.e. for any neighbourhood $V \ni f(x_0)$ (i.e. $f(x_0) \in \text{int}(V)$), $f^{-1}(V)$ is a neighbourhood of x_0. Fix $\varepsilon > 0$.

$$|f(x_0) - f(x)| \leq |f(x_0) - f_n(x_0)| + |f_n(x_0) - f_n(x)| + |f_n(x) - f(x)|$$
$$\leq 2\|f - f_n\|_{sup} + |f_n(x_0) - f_n(x)|.$$

Fix $n \in \mathbb{N}$ so that $\|f - f_n\|_{sup} < \varepsilon/3$. Since f_n is continuous at x_0, $\exists$ open $U \ni x_0$ such that $\forall x \in U$, we have $|f_n(x_0) - f_n(x)| < \varepsilon/3$. Thus, $\forall x \in U$, we have $|f(x_0) - f(x)| < 2 \cdot \varepsilon/3 + \varepsilon/3 = \varepsilon$. $\square$

When X is compact, $BC(X) = C(X)$ the space of all continuous functions on X.

(e) Again let X be a topological space. Let $C_0(X)$ denote the set of all $f \in C(X)$ such that $\forall \varepsilon > 0$ the set $\{|f| \geq \varepsilon\} := \{x \in X : |f(x)| \geq \varepsilon\}$ is compact. Note that such an f is bounded on $\{|f| \geq 1\}$ by compactness and of course $f|_{\{|f| < \varepsilon\}}$ is bdd by ε, so f is bdd, hence $C_0(X) \subseteq BC(X)$.

Furthermore, $C_0(X)$ is a subspace because
$$\{|f+g| \geq \varepsilon\} \subseteq \{|f| \geq \varepsilon/2\} \cup \{|g| \geq \varepsilon/2\}.$$
is a closed subset of a compact set (the set $\{|f+g| \geq \varepsilon\}$ is closed because it's the preimage under $|f+g|$ of the closed set $[\varepsilon, +\infty)$).

Claim. $C_0(X)$ is closed in $BC(X)$, hence a Banach space.

Proof. Let $f_n \to f$ in sup norm in $BC(X)$ where $(f_n) \subseteq C_0(X)$. Then $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n \geq N$, we have $\|f_n - f\| < \varepsilon/2$. In particular, for any $x \in X$, $|f(x)| \geq \varepsilon$ implies $|f_n(x)| \geq \varepsilon/2$, thus

$$\{|f| \geq \varepsilon\} \subseteq \{|f_n| \geq \varepsilon/2\},$$

hence is compact being a closed subset of a compact set. $\square$

(f) Let $(X, \mathcal{B})$ be a measurable space. Jordan decomposition theorem says that every signed measure ν on $(X, \mathcal{B})$ uniquely decomposes into a difference of two orthogonal unsigned measures $\nu = \nu_+ - \nu_-$, which are finite if ν is. Call the (unsigned) measure $|\nu| := \nu_+ + \nu_-$ the total variation (measure) of ν. It is an exercise (HW) to check that for each $B \in \mathcal{B}$,

$$|\nu|(B) = \sup_{\varphi} \left| \int_B \varphi \, d\nu \right|,$$

where φ ranges over all $\mathcal{B}$-measurable real-valued functions with $|\varphi| \leq 1$. Denoting by $M_{\mathbb{R}}(X, \mathcal{B})$ the $\mathbb{R}$-vector space of all finite signed measures on $(X, \mathcal{B})$, the function

$$\|\nu\|_{TV} := |\nu|(X) = \sup_{|\varphi| \leq 1} \left| \int_X \varphi \, d\nu \right|.$$

is a norm on $M_{\mathbb{R}}(X, \mathcal{B})$, called the total variation norm. This makes $M_{\mathbb{R}}(X, \mathcal{B})$ a normed $\mathbb{R}$-vector space.

Claim. $M_{\mathbb{R}}(X, \mathcal{B})$ is a real Banach space.

Proof. Uses the absolutely convergent series criterion and the Radon-Nikodym theorem. Left as a good HW exercise. $\square$

We also define the complex version: a finite complex measure on $(X, \mathcal{B})$ is a function

$$\rho : \mathcal{B} \to \mathbb{C}$$

which is of the form $\rho = \rho_{re} + i\rho_{im}$, where ρ_{re}, ρ_{im} are finite signed measures on $(X, \mathcal{B})$. Define the total variation (measure) of ρ by

$$|\rho|(B) = \left| |\rho_{re}|(B) + i|\rho_{im}|(B) \right| = \sqrt{|\rho_{re}|(B)^2 + |\rho_{im}|(B)^2}.$$

Again it's not hard to check (HW) that for each $B \in \mathcal{B}$, we have

$$|\rho|(B) = \sup_{\varphi} \left| \int \varphi \, d\rho \right|,$$

where φ ranges over all complex-valued $\mathcal{B}$-measurable functions with $|\varphi| \leq 1$. Then the total variation norm $\|\rho\|_{TV} := |\rho|(X)$ makes $M_{\mathbb{C}}(X, \mathcal{B})$ into a normed $\mathbb{C}$-vector space. It also follows from the real counterpart that $M_{\mathbb{C}}(X, \mathcal{B})$ is also a Banach space.